

THE PHASETRANSITION IN THE MULTIFLAVOUR SCHWINGER MODEL

S. DÜRR¹

¹ *Paul Scherrer Institut, Particle Theory Group, 5232 Villigen PSI, Switzerland*

A summary is given of a quantization of the multiflavour Schwinger model on a finite-temperature cylinder with chirality-breaking boundary conditions at its spatial ends, and it is shown that the analytic expression for the chiral condensate implies that the theory exhibits a second order phasetransition with $T_c=0$.

The (euclidean) Schwinger model (SM), i.e. QED(2) with one or several massless fermions, is defined through the generating functional

$$Z[\bar{\eta}, \eta] \sim \int D(A, \bar{\psi}, \psi) e^{-\int \frac{1}{4} F^2 - \int \bar{\psi} \not{D} \psi + \int (\bar{\eta} \psi + \bar{\psi} \eta)} \quad (1)$$

where the integration is over all smooth fields on the manifold M , e.g. a torus $T_{L,\beta} = [0, L] \times [0, \beta]$. The theory allows for a finite-temperature interpretation if bosonic (fermionic) fields are (anti-)periodic in the second direction and $\beta \ll L$. From (1) one may proceed to compute an observable

$$\langle O \rangle_{L,\beta} = \frac{1}{Z[0,0]} \int D(\text{fields}) O(\text{fields}) e^{-S(\text{fields})} \quad (2)$$

and study its behaviour at various box-lengths L and temperatures $T=1/\beta$.

From a mathematical point of view the choice $M = T_{L,\beta}$ looks appealing, since the finite volume provides a natural infrared regularization, gauge-fields fall into different topological classes labeled by $\nu = \frac{g}{4\pi} \int F_{\mu\nu} \epsilon_{\mu\nu}$, and the index-theorem holds true [1]: The number of exact zero-modes of the Dirac operator on a given background A equals $|\nu|$, and the nonzero modes come in pairs $(\not{D}\psi_{\pm} = \pm i\lambda\psi_{\pm})$ of opposite chirality $(\gamma_5\psi_{\pm} = \pm\psi_{\pm})$, i.e. the diagonal entries (in a chiral representation) of the Green's function on the subspace orthogonal to the zero-modes vanish: $S'_{\pm\pm} = \text{tr}(P_{\pm}S') = 0$, where $P_{\pm} = \frac{1}{2}(1 \pm \gamma_5)$. This and the form (1) takes after the fermions have been integrated out [2]

$$Z[\bar{\eta}, \eta] \sim \sum_{\nu \in \mathbf{Z}} \int D A^{(\nu)} e^{-\int \frac{1}{4} F^2} \prod_{k=1}^{N_f |\nu|} (\bar{\eta} \psi_k)(\bar{\psi}_k \eta) \det'(\not{D})^{N_f} e^{\int \bar{\eta} S' \eta} \quad (3)$$

(the product is over the zero-modes !) imply for the chiral condensate on the torus

$$\langle \bar{\psi} P_{\pm} \psi \rangle_{L,\beta} = \frac{\delta^2}{\delta \bar{\eta}_{\pm} \delta \eta_{\pm}} \Big|_{\bar{\eta}, \eta=0} \log Z[\bar{\eta}, \eta] \begin{cases} \neq 0 & (N_f=1) \\ = 0 & (N_f \geq 2) \end{cases} \quad (4)$$

In the singleflavour case the well-known (anomaly induced) Schwinger value $ge^\gamma/4\pi^{3/2}$ is reproduced for $L, \beta \rightarrow \infty$. The problem is that in the multiflavour theory the condensate on the torus is zero already at finite L, β and not only in the limit $L\beta \rightarrow \infty$ (where it has to vanish, due to Coleman's theorem [3]); this makes the torus, from a physical point of view, an uninteresting manifold.

Quite generally in field theory and statistical mechanics, if one wants to establish that a system breaks a global symmetry spontaneously (or investigate how it avoids doing so), one *needs to break the symmetry explicitly* and observe how the system behaves as the symmetry breaking source is *turned off*. A straightforward way of doing this, both in QCD and the SM, is to give the fermions a mass m . The problem with this choice, in the case of the SM, is that it destroys exact analytic tractability¹. Alternatively, one may break the chiral symmetry by boundary conditions and study implications of sending the boundaries to infinity [6]. The idea is to quantize the SM on a thermal cylinder $[0, L] \times [0, \beta]$, where at the two ends $x^1=0, L$ (no identification) fields are subject to a member of a one-parameter family of local linear boundary conditions (labeled by $\theta \in \mathbf{R}$). Peculiar properties of this choice include [6]:

- No $U(1)_V$ current leaks through the boundary, i.e. $j_\perp = \psi^\dagger \not{n} \psi = 0$ on ∂M .
- The topological charge is *not quantized*, i.e. $\nu = \frac{g}{4\pi} \int \epsilon_{\mu\nu} F_{\mu\nu} = \frac{g}{2\pi} \int E \in \mathbf{R}$.
- There is no pairing and Dirac modes have no fixed chirality, i.e. $(S_\theta)_{\pm\pm} \neq 0$.
- The boundary condition (θ) adds a P/CP-odd piece to the effective action²

$$\Gamma = \frac{1}{2g^2} \int \phi(\Delta^2 - N_f \frac{g^2}{\pi} \Delta) \phi + \theta \frac{N_f}{2\pi} \int \Delta \phi + N_f \Gamma(c), \quad (5)$$

¹Bosonization rules [4] allow to separate the theory into a heavy (above the mass-gap) and a light (below) sector, and recently an exact solution for the latter has appeared [5].

²Note that in 2 dimensions the coupling g has the dimension of a mass.

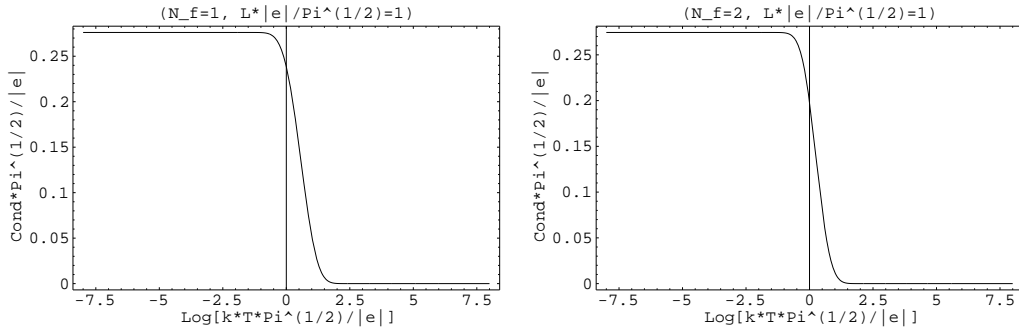


Figure 1: Crossover-phenomenon of the dimensionless condensate $|\langle \bar{\psi} P_{\pm} \psi \rangle|/\mu$ as a function of $\log(kT/\mu)$ at fixed $L=1/\mu$ for $N_f=1$ (left) and $N_f=2$ (right).

where the Hodge decomposition $gA_\mu = -\epsilon_{\mu\nu}\partial_\nu\phi + \partial_\mu\chi + \frac{2\pi}{\beta}c\delta_{\mu 2}$ has been used.

- There are no fermion zero-modes, hence (3) is replaced by the simpler form

$$Z[\bar{\eta}, \eta] \sim \int D\phi dc e^{-\Gamma} e^{\bar{\eta} S_\theta \eta}. \quad (6)$$

For $N_f = 1$ and $N_f = 2$ the resulting functional integral may be evaluated completely [7]. Specializing to midpoints ($\xi = \frac{L}{2}$), the condensate for $N_f = 1$ is

$$\begin{aligned} \frac{\langle \bar{\psi} P_\pm \psi \rangle}{\mu} &= \pm \frac{e^{\pm\theta/\cosh(\lambda/2)}}{4\pi} \left(1 + 2 \sum_{n \geq 1} (-1)^n \frac{1}{\cosh(n\pi\tau)} \exp(-n^2\pi\tau) \right) \cdot \\ &\quad \exp \left\{ \gamma - 2 \sum_{j \geq 1} (-1)^j K_0(j\lambda) \right\}. \quad (\tau \gg 1) \\ &\quad \exp \left\{ \sum_{n \geq 0} \frac{4}{(2n+1)(e^{2(2n+1)\pi\tau} - 1)} - ((2n+1) \rightarrow \sqrt{(2n+1)^2 + (\frac{\lambda}{\pi})^2}) \right\} \\ \frac{\langle \bar{\psi} P_\pm \psi \rangle}{\mu} &= \pm \frac{e^{\pm\theta/\cosh(\lambda/2)}}{4\pi} \sum_{m \geq 0} (-1)^m \frac{e^{-\pi((2m+1)^2 - 1)/4\tau}}{\sinh(\pi(2m+1)/2\tau)} \times \\ &\quad \sum_{k \geq 0} \cosh\left(\frac{\pi(2m+1)(2k+1)}{2\tau}\right) (\text{erf}(\frac{(k+1)\sqrt{\pi}}{\sqrt{\tau}}) - \text{erf}(\frac{k\sqrt{\pi}}{\sqrt{\tau}})) \cdot \\ &\quad \exp \left\{ \gamma + \frac{\pi(1-\tanh(\lambda/2))}{\sigma} - 2 \sum_{j \geq 1} K_0(j\sigma) \right\}. \quad (\tau \ll 1) \\ &\quad \exp \left\{ - \sum_{m \geq 1} \frac{2}{m(e^{m\pi/\tau} + 1)} - (m \rightarrow \sqrt{m^2 + (\frac{\sigma}{2\pi})^2}) \right\} \end{aligned} \quad (7)$$

while for $N_f = 2$ the result reads

$$\begin{aligned} \frac{\langle \bar{\psi} P_\pm \psi \rangle}{\mu} &= \pm \frac{2^{1/4} e^{\pm\theta/\cosh(\lambda/\sqrt{2})}}{4\sqrt{\pi}\sqrt{\lambda}} \cdot \\ &\quad \left(1 + 2 \sum_{n \geq 1} (-1)^n \frac{e^{-n^2\pi\tau/2}}{\cosh(n\pi\tau)} \cdot \frac{\sum_{k \in \mathbf{Z}} \frac{e^{-(n/2-k)^2 2\pi\tau} + e^{-(n/2+k)^2 2\pi\tau}}{2}}{\sum_{k \in \mathbf{Z}} e^{-2k^2\pi\tau}} \right) \cdot \\ &\quad \exp \left\{ \frac{\gamma}{2} - \sum_{j \geq 1} (-1)^j K_0(j\sqrt{2}\lambda) \right\}. \quad (\tau \gg 1) \\ &\quad \exp \left\{ \sum_{n \geq 0} \frac{2}{(2n+1)(e^{2(2n+1)\pi\tau} - 1)} - ((2n+1) \rightarrow \sqrt{(2n+1)^2 + 2(\frac{\lambda}{\pi})^2}) \right\} \\ \frac{\langle \bar{\psi} P_\pm \psi \rangle}{\mu} &= \pm \frac{2^{1/4} e^{\pm\theta/\cosh(\lambda/\sqrt{2})}}{4\sqrt{\pi}\sqrt{\sigma}} \sum_{m \geq 0} (-1)^m \frac{e^{-\pi((2m+1)^2 - 1)/8\tau}}{\sinh(\pi(2m+1)/2\tau)} \times \\ &\quad \frac{\sum_{q \in \mathbf{Z}} e^{-\pi q^2/2\tau} \sum_{p \geq 0} e^{(-1)^{p+q+m} \frac{\pi(p+1/2)(m+1/2)}{\tau}} (\text{erf}(\frac{(p+3/2)}{\sqrt{2\tau/\pi}}) - \text{erf}(\frac{(p-1/2)}{\sqrt{2\tau/\pi}}))}{\sum_{q \in \mathbf{Z}} e^{-\pi q^2/2\tau}} \cdot \\ &\quad \exp \left\{ \frac{\gamma}{2} + \frac{\pi(1-\tanh(\lambda/\sqrt{2}))}{2\sqrt{2}\sigma} - \sum_{j \geq 1} K_0(j\sqrt{2}\sigma) \right\}. \quad (\tau \ll 1) \\ &\quad \exp \left\{ - \sum_{m \geq 1} \frac{1}{m(e^{\pi m/\tau} + 1)} - (m \rightarrow \sqrt{m^2 + 2(\frac{\sigma}{2\pi})^2}) \right\} \end{aligned} \quad (8)$$

where $\mu = \frac{g}{\sqrt{\pi}}$ and $\tau = \frac{\beta}{2L}$, $\sigma = \mu\beta$, $\lambda = \mu L$. In either case a representation which converges fast for $\tau \gg 1$ and another one which converges fast for $\tau \ll 1$ is given. Note however, that for finite L, β the two representations agree exactly.

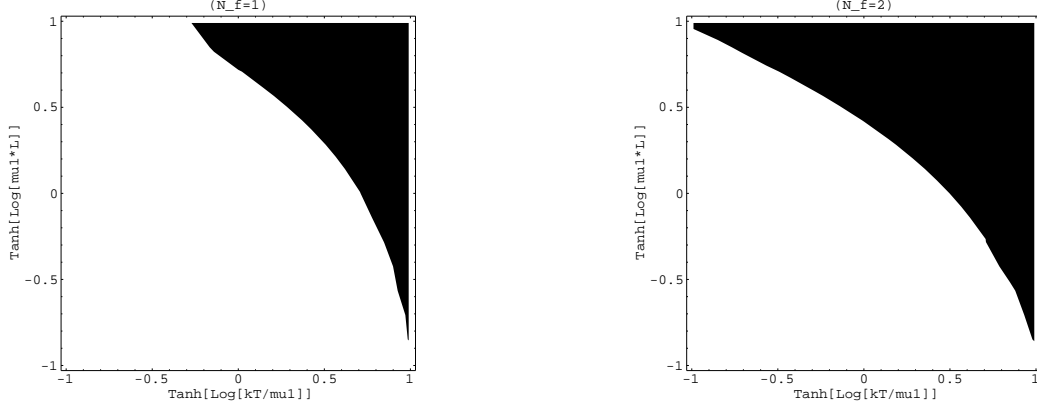


Figure 2: Quasi-Phasestructure in the $\log(kT/\mu)$ - $\log(L\mu)$ plane for $N_f = 1$ and $N_f = 2$. White points: $\langle \bar{\psi}\psi \rangle / \mu > e^\gamma / 8\pi$. Black points: $\langle \bar{\psi}\psi \rangle / \mu < e^\gamma / 8\pi$.

A numerical evaluation of (7) and (8) – originally undertaken just to check the latter assertion – gives a picture as shown in Fig. 1: For any finite L there is a low-temperature regime where the combination of anomalous ($N_f = 1$) or attempted spontaneous ($N_f = 2$) and explicit symmetry breaking generates a non-zero condensate, and there is a “critical temperature” where the condensate decays through a fairly well localized “symmetry quasi-restoration” process to a value exponentially close (but not equal) to zero. Repeating the exercise for various L values [7], one realizes that crossover-temperature and height of the plateau increase with diminishing L both for $N_f = 1$ and 2, whereas for $L\mu \rightarrow \infty$ remarkable differences show up: For $N_f = 1$ kink-position and plateau-height are fairly stable, while for $N_f = 2$ the kink keeps moving left and the plateau-height decreases unlimitedly. It makes thus sense to introduce the concept of a *quasi-phasestructure* [7] in order to distinguish points in parameter-space where the condensate is *manifestly non-zero* (white in Fig. 2) from those where the chiral symmetry is *quasi-restored* (black in Fig. 2). The point $\beta = L = \infty$ (upper left corner in both plots) is clearly in the broken (quasi-)phase for $N_f = 1$, while it seems to lie near or right on the crossover-line for $N_f = 2$ (note that areas close to the boundaries in Fig. 2 are cut off for numerical reasons). A closer look at (7, 8) reveals that indeed the one-flavour condensate approaches its (finite) value at $\beta = L = \infty$ smoothly from any direction (within the plot), whereas in the two-flavour theory the condensate *vanishes differently, depending on which limit is performed first* [7]:

$$\langle \bar{\psi}\psi \rangle \sim \begin{cases} 1/\sqrt{\lambda} & (N_f = 2, T = 0) \\ e^{-\text{const } \lambda} & (N_f = 2, T > 0) \end{cases} \quad . \quad (9)$$

Of course this “smells” like a critical phenomenon, but in order to make contact with the general theory, we need to convert our result to an expression where the symmetry is broken in the bulk, i.e. by a fermion mass term. The naive (dimensionally motivated) identification $L^{-1} \leftrightarrow m$ then leads to ($N_f \geq 2$)

$$\langle \bar{\psi}\psi \rangle \sim \begin{cases} m^{(N_f-1)/N_f} & (T=0) \\ e^{-\text{const}/m} & (T>0) \end{cases} \implies \chi = \frac{d}{dm} \langle \bar{\psi}\psi \rangle \sim \begin{cases} m^{-1/N_f} & \rightarrow \infty \\ \frac{e^{-\text{const}/m}}{m^2} & \rightarrow 0 \end{cases}, \quad (10)$$

where we have generalized to arbitrary $N_f \geq 2$. If (10) were correct, it would indicate a second order phasetransition with critical temperature $T_c=0$. The problem is that the corresponding critical coefficient δ in $\langle \bar{\psi}\psi \rangle \sim m^{1/\delta}$ (at the transition) would take the value $N_f/(N_f-1)$, which is at variance with the correct (bosonization rule based) result $\delta=(N_f+1)/(N_f-1)$ [8].

The problem can be solved by realizing that the entire manifold $[0, L] \times [0, \beta]$ our calculation was based on, may be seen as a model of the inside of a two-dimensional meson at finite temperature (the chirality breaking boundary conditions at $x^1=0, L$ do indeed resemble the MIT bag boundary conditions), hence the proper identification is $L^{-1} \leftrightarrow M$, where $M \sim m^\#$ is the mass of the lightest (pseudoscalar) meson. Unlike in QCD, the exponent $\#$ depends on the number of active flavours. The simplest way to get the value $\#$ is the following: *Require* that the multiflavour SM satisfies, in complete analogy with QCD, a Gell-Mann – Oakes – Renner type PCAC-relation³ (for $T=0, N_f \geq 2$)

$$m \langle \bar{\psi}\psi \rangle \sim M^2, \quad (11)$$

from which, upon using $\langle \bar{\psi}\psi \rangle \sim (1/L)^{(N_f-1)/N_f} \sim (m^\#)^{(N_f-1)/N_f}$, one concludes

$$1 + \# \frac{N_f-1}{N_f} = 2\# \implies \# = \frac{N_f}{N_f+1}. \quad (12)$$

Hence, using the relationship $L^{-1} \leftrightarrow m^{N_f/(N_f+1)}$, the corrected version of (10) is

$$\langle \bar{\psi}\psi \rangle \sim \begin{cases} m^{(N_f-1)/(N_f+1)} & (T=0) \\ e^{-\text{const}/m^{N_f/(N_f+1)}} & (T>0) \end{cases} \implies \chi \sim \begin{cases} m^{-2/(N_f+1)} & \rightarrow \infty \\ \frac{e^{-\text{const}/m^{N_f/(N_f+1)}}}{m^{1+N_f/(N_f+1)}} & \rightarrow 0 \end{cases}, \quad (13)$$

from which we see that the multiflavour SM shows indeed a second order phasetransition with zero critical temperature and the critical exponent δ , defined through $\langle \bar{\psi}\psi \rangle(T=T_c) \sim m^{1/\delta}$, is

$$\delta = \frac{N_f+1}{N_f-1}, \quad (14)$$

³Admittedly, this is done with an eye on the bosonization approach, where (11) was first *derived* from the proper scaling of $\langle \bar{\psi}\psi \rangle$ and M [9].

as originally derived by Smilga and Verbaarschot [8].

We shall conclude with two comments:

(i) The SM feature worth emphasizing is the analogy with QCD: For $N_f = 1$ either theory shows a smooth crossover and the nonzero chiral condensate does not indicate SSB, since the symmetry is already broken by the anomaly. For $N_f \geq 2$ and with a small symmetry breaking source there is a striking similarity to QCD *slightly above the phasetransition*: The Polyakov loop in the multiflavour SM is real and positive, and the chiral condensate is almost zero; the system “tries” to break the axial flavour symmetry spontaneously, the spectrum shows a “mass gap” between the “Schwinger particle” with mass $g\sqrt{\frac{N_f}{\pi}} + O(m)$ (which is the analogue of the η') and the $N_f^2 - 1$ light “Quasi-Goldstones” with mass $M \sim m^{N_f/(N_f+1)}$ [4, 9]. The latter get sterile in the chiral limit (as required by Coleman’s theorem [3]), but the important point is that, as long as one stays away from the chiral limit, these “pions” dominate the long-range Green’s functions between external (S,P,V,A)-currents.

(ii) The sketch presented above exemplifies the glory and the misery of a path-integral quantization of a soluble model with non-standard boundary conditions: Symmetry breaking boundary conditions prove useful to force a field theory into a definite groundstate and help, for this reason, to explore systems which successfully show SSB or attempt doing so. On the other hand, the path-integral approach is not very transparent; formulas may be clumsy (cf. (7, 8)) and one has no clue what are the relevant physical degrees of freedom. It is therefore little surprise that the fact that the multiflavour SM shows a second order phasetransition with zero critical temperature (the only T_c possible in 2 dimensions) was first derived in the bosonized approach [4, 8].

- [1] H. Joos, Helv. Phys. Acta 63, 670 (1990).
- [2] I. Sachs, A. Wipf, Helv. Phys. Acta 65, 652 (1992).
- [3] S. Coleman, Comm. Math. Phys. 31, 259 (1973).
- [4] S. Coleman, Phys. Rev. D 11, 2088 (1975).
- [5] Al.B. Zamolodchikov, Int. J. Mod. Phys. A 10, 1125 (1995).
- [6] P. Hrasko, J. Balog, Nucl. Phys. B 245, 118 (1984); A. Wipf, S. Dürr, Nucl. Phys. B 443, 201 (1995); S. Dürr, A. Wipf, Ann. Phys. 255, 333 (1997).
- [7] S. Dürr, Ann. Phys. 273, 1 (1999).
- [8] A.V. Smilga, J.J.M. Verbaarschot, Phys. Rev. D 54, 1087 (1996).
- [9] A.V. Smilga, Phys. Lett. B 278, 371 (1992).